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THE APPLICATION OF THE GOVERNING PRINCIPLE OF DISSIPATIVE PROCESSES TO BÉNARD CONVECTION

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Abstract—The paper deals with the application of the Governing Principle of Dissipative Processes (GPDP) to the Bénard convection, that is, the motion arising from the thermal instability of a thin horizontal layer of fluid when a steady temperature contrast is maintained across it. The linearized equations of Bénard convection are derived with the help of the universal form of the governing principle. The so-called dual field method developed on the basis of GPDP is used to get the solution of the problem in all the three representations of the principle, namely, in universal, force and flux respectively. It is found that the results obtained by the actual variational principle, that is; by the universal form of the principle differ by less than 2·5 per cent with those of exact values while flux representation gives the best approximate results and the force representation is found to be the worst one.

NOMENCLATURE

a, $= \sqrt{(a_1^2+a_2^2)}$, wave number;
a_c, critical wave number;
a_i, specific value of the continuum referred to unit mass;
A, B, variational parameters;
c_v, specific heat at constant volume;
d, thickness of the layer;
g, acceleration due to gravity;
$g_i(x)$, linearly independent functions;
G, trial function for velocity;
$\mathbf{T}_i$, thermodynamic current density;
$\mathbf{T}_q$, heat current density;
k, thermal diffusivity;
L_{ik}, phenomenological coefficients;
$\mathbf{n}$, unit vector;
p, hydrostatic pressure;
$\mathbf{P}$, pressure tensor;
$\overset{\circ}{\mathbf{P}}_{vs}$, symmetrical part of pressure tensor;
R, Rayleigh number;
R_c, critical Rayleigh number;
t, time;
T, perturbed temperature field;
T^*, approximate temperature field;
$\mathbf{v}$, perturbed velocity field;
$\mathbf{v}^*$, approximate velocity field;
v_1, v_2, v_3, velocity components in x_1, x_2 and x_3 directions respectively;
v_1^*, v_2^*, v_3^*, approximate velocity components;
V, volume of the continuum;
X_i, thermodynamic forces;
z, $= x_3/d$, co-ordinate in dimensionless form.

Greek symbols

α, coefficient of volume expansion;
$\alpha_i(t)$, variational parameters;
β, temperature gradient;
λ, thermal conductivity;
μ, coefficient of viscosity;
ν, kinematic viscosity;
ω, frequency of oscillations;
θ, trial function for temperature field;
ρ_0, density of the continuum;
Ω, surface of the system;
δ_{ik}, Kronecker symbol;
Γ_i, state parameters;
Γ_i^*, approximate value of Γ_i;
σ, entropy production;
ψ, ϕ, dissipation potentials;
σ_i, source density per unit volume and time.

1. INTRODUCTION

IN 1965, a new variational principle was proposed by Gyarmati [1, 2]—which has some fundamental relationship with the Onsager's principle of least dissipation of energy [3, 4]—by means of which dissipative transport processes in space and time can be described.

The so-called Gaussian type of universal form of the principle seemed to be extremely general—due to its validity in quasi-linear and in certain types of non-linear cases—and was, therefore, called "The Governing Principle of Dissipative Processes" [5, 6].

The most general form of the Gyarmati's principle is represented by the following equality [5–7]

$$\delta \int_V [\sigma - \psi - \phi]\, \mathrm{d}V = 0, \tag{1}$$

for any instant of time under constraints that the balance equations

$$\rho \dot{a}_i + \nabla . \mathbf{T}_i = \sigma_i, \quad (i = 1, 2, \ldots f) \tag{2}$$

are satisfied. Here σ is the entropy production of the system, ψ and ϕ are the local dissipation potentials and the integration is considered over the total volume, V, of the continuum. $\mathbf{T}_i$ is the substantial current density and σ_i is the source of a_i per unit volume and unit time. The entropy production, σ, in the case of irreversible processes taking place in a continuum can always be written in the following bilinear form

$$\sigma = \sum_{i=1}^{f} \mathbf{T}_i . \mathbf{X}_i \geqslant 0, \tag{3}$$

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where $\mathbf{T}_i$ and $\mathbf{X}_i$ are the thermodynamic currents and forces respectively and the quantity σ is positive definite one according to the second law of thermodynamics. According to the experimentally well-verified Onsager theory of thermodynamics, the currents are linear functions of the forces, i.e.

$$\mathbf{T}_i = \sum_{k=1}^{f} L_{ik}\,\mathbf{X}_k, \quad (i = 1,2,\ldots f), \tag{4}$$

where the coefficients, L_{ik}, are constants and represent the conductivities. These relations are called the linear kinematical constitutive laws for the coefficients of which the famous Onsager reciprocal relations

$$L_{ik} = L_{ki}, \quad (i,k = 1,2,\ldots f); \tag{5}$$

are valid.

The constitutive equations (4) can alternatively be written as

$$\mathbf{X}_i = \sum_{k=1}^{f} R_{ik}\,\mathbf{T}_k, \quad (i = 1,2,\ldots f), \tag{6}$$

where the coefficients R_{ik} represent the resistances. The matrices of the conductivities and resistances are mutually reciprocal, i.e.

$$\sum_{m=1}^{f} L_{im} R_{mk} = \sum_{m=1}^{f} R_{im} L_{mk} = \delta_{ik}, \quad (i,k = 1,2,\ldots f). \tag{7}$$

Here δ_{ik} is the Kronecker symbol, i.e. $\delta_{ik} = 1$, $i = k$ and $\delta_{ik} = 0$, $i \neq k$. Thus R_{ik} satisfy the reciprocal relations

$$R_{ik} = R_{ki}, \quad (i,k = 1,2,\ldots f). \tag{8}$$

The local dissipation potentials ψ and ϕ are defined in the following homogeneous quadratic forms

$$\psi(\mathbf{X},\mathbf{X}) \equiv \frac{1}{2}\sum_{i,k=1}^{f} L_{ik}\,\mathbf{X}_i.\mathbf{X}_k \geqslant 0, \tag{9}$$

$$\phi(\mathbf{T},\mathbf{T}) \equiv \frac{1}{2}\sum_{i,k=1}^{f} R_{ik}\,\mathbf{T}_i.\mathbf{T}_k \geqslant 0, \tag{10}$$

which correspond to the entropy form (3) written in terms of thermodynamic forces and currents respectively.

The principle (5) with (3), (9) and (10) takes the form

$$\delta \int_V \left[\sum_{i=1}^{f} \mathbf{T}_i.\nabla\Gamma_i - \frac{1}{2}\sum_{i,k=1}^{f} L_{ik}\nabla\Gamma_i.\nabla\Gamma_k - \frac{1}{2}\sum_{i,k=1}^{f} R_{ik}\,\mathbf{T}_i.\mathbf{T}_k \right] \mathrm{d}v = 0. \tag{11}$$

In (11) the thermodynamic forces, $\mathbf{X}_i$, are substituted by $\nabla\Gamma_i$, which are given as

$$\mathbf{X}_i = \nabla\Gamma_i, \tag{12}$$

since in the case of transport processes the forces can always be generated as the gradients of certain "Γ" variables which are state parameters and simultaneously internal parameters with respect to the forces [5, 6].

2. THE DUAL FIELD METHOD

The approximate method which we are going to discuss has its base on the following facts of the GPDP:

1. The entropy production

$$\sigma = \sum_{i=1}^{f} \mathbf{T}_i.\nabla\Gamma_i$$

is a symmetrical bilinear expression of current densities $\mathbf{T}_i$ and the conjugated forces $\nabla\Gamma_i$.

2. The local dissipation potentials

$$\psi \equiv \frac{1}{2}\sum_{i,k=1}^{f} L_{ik}\nabla\Gamma_i.\nabla\Gamma_k \text{ and } \phi \equiv \frac{1}{2}\sum_{i,k=1}^{f} R_{ik}\,\mathbf{T}_i.\mathbf{T}_k$$

are connected with one-another by Legendre dual transformation with respect to the current densities $\mathbf{T}_i$ and the conjugated forces $\nabla\Gamma_i$.

3. The structure of the varied form of the principle (11) possesses the duality property with respect to the transport equations and the linear constitutive laws. From this volume integral the transport equations and the linear constitutive laws are following simultaneously.

These facts ensure the possibility of developing the approximate method which may be called "The Dual Field Method" [8].

The two sets of independent variables $[\mathbf{T}_1,\mathbf{T}_2,\ldots\mathbf{T}_f]$ and $[\nabla\Gamma_1,\nabla\Gamma_2,\ldots\nabla\Gamma_f]$ are connected with each other by the relations (4) and (6). In this method, therefore, we assume one set of these variables and then the other set can be obtained with the help of the constitutive relations. In the irreversible transport phenomena, the variables Γ_i are fundamental ones, since their gradients, $\nabla\Gamma_i$, are the driving forces of dissipative transport processes. We, thus, approximate the set $[\nabla\Gamma_1,\nabla\Gamma_2,\ldots\nabla\Gamma_f]$ by another set $[\nabla\Gamma_1^*,\nabla\Gamma_2^*,\ldots\nabla\Gamma_f^*]$. Consequently the corresponding current densities are obtained by the following constitutive equations

$$\mathbf{T}_i = \sum_{k=1}^{f} L_{ik}(\Gamma_1,\Gamma_2,\ldots\Gamma_f)\nabla\Gamma_k^*, \quad (i = 1,2,\ldots f). \tag{13}$$

It is interesting that the duality property of the governing principle is preserved and in the case of exact solution the two sets of fundamental variables Γ_i and Γ_i^* coincide, i.e. $\Gamma_i \equiv \Gamma_i^*$ for all i.

The principle (11) with the help of (13) takes the following form

$$\delta \int_V \sum_{i,k=1}^{f} \left[-\tfrac{1}{2}L_{ik}(\nabla\Gamma_i - \nabla\Gamma_i^*).(\nabla\Gamma_i - \nabla\Gamma_i^*)\right]\mathrm{d}V = 0, \tag{14}$$

which together with the balance equations

$$\rho\dot{a}(\Gamma_i) + \sum_{k=1}^{f} \nabla.(L_{ik}\nabla\Gamma_k^*) = \sigma(\Gamma_i), \quad (i = 1,2,\ldots f) \tag{15}$$

serves the basis for the dual field method.

We shall confine our following treatment for the case of one, Γ, parameter. In this case, the principle (14) is obtained as

$$\delta \int_V \left[-\tfrac{1}{2}L(\nabla\Gamma - \nabla\Gamma^*).(\nabla\Gamma - \nabla\Gamma^*)\right]\mathrm{d}V = 0, \tag{16}$$

and the balance equations (15) become

$$\rho\dot{a}(\Gamma)+\nabla.(L\nabla\Gamma^*)=\sigma(\Gamma), \tag{17}$$

where L is the conductivity of the system. To get the approximate solution, we assume the field, Γ, in the form

$$\Gamma\equiv\Gamma^{(n)}=\sum_{i=1}^{n}\alpha_i(t)g_i(\mathbf{x}), \tag{18}$$

where $[g_i(\mathbf{x})]_{i=1}^n$ are a set of linearly independent functions which satisfy the boundary conditions imposed on Γ. The coefficients, $\alpha_i(t)$, are the variational parameters to be determined with the help of the principle (16). The balance equation (17), with the help of (18), results to

$$\rho(\Gamma^{(n)})\frac{\partial a(\Gamma^{(n)})}{\partial t}+\nabla.[L(\Gamma^{(n)})\nabla\Gamma^{*(n)}]=\sigma(\Gamma^{(n)}). \tag{19}$$

This balance equation serves the purpose to determine the field Γ^* in the form $\Gamma^{*(n)}$ and the solution of (19) with the appropriate boundary conditions may be obtained as

$$\Gamma^*\equiv\Gamma^{*(n)}=\Gamma^{*(n)}(\mathbf{x},t,\alpha_1,\dots\alpha_n,\dot{\alpha}_1\dots\dot{\alpha}_n). \tag{20}$$

It is remarkable that the volume integral (3) is maximum at any instant of time for the real physical processes, that is; for the exact value of the parameters Γ_i and the current densities $\mathbf{T}_i$. It is fundamentally important that the maximum is zero for any time [5]. It is found, however, that in the application of approximate procedure, the volume integral generally becomes a function of time and therefore the volume integral may be integrated over the time interval $0<t<\infty$ during which the process is considered. Thus the principle (16) becomes

$$\delta\int_0^\infty\int_V[-\tfrac{1}{2}L(\nabla\Gamma-\nabla\Gamma^*).(\nabla\Gamma-\nabla\Gamma^*)]\times dV\,dt=0. \tag{21}$$

The total variation of the principle (21) with the values of Γ and Γ^* from (18) and (20) becomes

$$\begin{aligned}&\int_V\left\{\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}\right.\\&\quad+\tfrac{1}{2}(\nabla\Gamma^{(n)}-\nabla\Gamma^{*(n)})\frac{\partial L^{(n)}}{\partial\Gamma^{(n)}}\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}\\&\quad+\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{*(n)}}{\partial\alpha_i}\\&\quad-\frac{d}{dt}\left[\left(\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right)\right.\\&\quad\left.\left.\times\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\right]\right\}dV+\oint_\Omega\left\{[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}\right.\\&\quad+[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{*(n)}}{\partial\alpha_i}-\frac{d}{dt}\\&\quad\left.\times\left[(L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)})\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\right]\right\}.d\boldsymbol{\Omega}=0,\\&\qquad(i=1,2\dots n);\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\int_V\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\Bigg|_{t=\infty}dv\\&\quad+\oint_\Omega[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\Bigg|_{t=\infty}.d\boldsymbol{\Omega}=0,\\&\qquad(i=1,2\dots n).\end{aligned} \tag{23}$$

In (23), the subscript denotes that the parameters, α_i, are evaluated at the moment $t=\infty$. Taking into consideration the transversality conditions (23) and the given initial conditions, we can solve the second-order partial differential equations (22) to get the parameters α_i and thus the fields $\Gamma^{(n)}$ and $\Gamma^{*(n)}$ respectively.

We have discussed the dual field method in universal forms of the principle. In the following we consider this method in the two partial forms of the principle as well.

(i) *The Dual Field Method in force representation*

In this case the variation is considered only with respect to the actual field, $\Gamma^{(n)}$, keeping the assumed field, $\Gamma^{*(n)}$, fixed, i.e. $\delta\Gamma^{*(n)}=0$. Taking this condition into consideration, (22) yields to

$$\begin{aligned}&\int_V\left\{\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}\right.\\&\quad\left.+\tfrac{1}{2}[\nabla\Gamma^{(n)}-\nabla\Gamma^{*(n)}]\frac{\partial L^{(n)}}{\partial\Gamma^{(n)}}\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}\right\}dV\\&\quad+\oint_\Omega[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{(n)}}{\partial\alpha_i}.d\boldsymbol{\Omega}=0,\\&\qquad(i=1,2\dots n).\end{aligned} \tag{24}$$

Here the parameters α_i can be determined from the set of equations (24) and hence $\Gamma^{(n)}$ and $\Gamma^{*(n)}$ are obtainable.

(ii) *The Dual Field Method in flux representation*

This representation is characterized by the condition $\delta\Gamma^{(n)}=0$ and can be used only if $\Gamma^{*(n)}$ contains variational parameters α_i and $\dot{\alpha}_i$. In this case the relations (22) and (23) reduce to

$$\begin{aligned}&\int_V\left\{\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{*(n)}}{\partial\alpha_i}\right.\\&\quad\left.-\frac{d}{dt}\left[\left(\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right)\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\right]\right\}dV\\&\quad+\oint_\Omega\left\{[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{*(n)}}{\partial\alpha_i}-\frac{d}{dt}\right.\\&\quad\left.\times\left[(L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)})\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\right]\right\}.d\boldsymbol{\Omega},\\&\qquad(i=1,2\dots n);\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\int_V\left[\rho^{(n)}\frac{\partial a^{(n)}}{\partial t}+\nabla.(L^{(n)}\nabla\Gamma^{(n)})-\sigma^{(n)}\right]\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\Bigg|_{t=\infty}dV\\&\quad+\oint_\Omega[L^{(n)}\nabla\Gamma^{(n)}-L^{(n)}\nabla\Gamma^{*(n)}]\frac{\partial\Gamma^{*(n)}}{\partial\dot{\alpha}_i}\Bigg|_{t=\infty}.d\boldsymbol{\Omega}=0,\\&\qquad(i=1,2\dots n).\end{aligned} \tag{26}$$

The solution of the equations (25), with the transversality conditions (26) and the initial conditions, give the variational parameters α_i.

We shall apply these methods to get the solution of the Bénard convection.

3. THE FORMULATION OF GPDP FOR BÉNARD CONVECTION

The balance equations of energy and momentum for the linearized Bénard convection are [9]

$$c_v \rho_0 \frac{\partial T}{\partial t} + \nabla . \mathbf{T}_q = c_v \rho_0 \beta \mathbf{n} . \mathbf{v}, \tag{27}$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla . \mathbf{P} = \rho_0 g \alpha \mathbf{n} T, \tag{28}$$

where $\mathbf{P}$ is the pressure tensor and is given by

$$\mathbf{P} = \delta p + \hat{\mathbf{P}}^{vs}, \tag{29}$$

p being the hydrostatic pressure and $\hat{\mathbf{P}}^{vs}$ is the symmetrical part of the pressure tensor whose trace is zero. $\mathbf{n} = (0, 0, 1)$ is the unit vector, $\mathbf{v}$ and T are the perturbation velocity and temperature fields respectively and c_v denotes the specific heat at constant volume while ρ_0 is the density of the continuum. α, β and g denote the coefficient of volume expansion, temperature gradient and the gravitational force respectively. It is well-known [5, 6] that in the formulation of Gyarmati's principle for thermohydrodynamical system, it is preferable to use the energy picture of the principle. For the formulation of the Bénard convection it is also convenient to use $\ln T$ instead of T in the balance equations (27) and (28)

$$c_v \rho_0 \frac{\partial}{\partial t} \ln T + \nabla . \mathbf{T}_q = c_v \rho_0 \beta \mathbf{n} . \mathbf{V}, \tag{30}$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla . \mathbf{P} = \rho_0 g \alpha \mathbf{n} \ln T. \tag{31}$$

Since we shall use the energy picture of the principle instead of the entropy picture, we shall consider the expression for energy dissipation, $T\sigma$, instead of the entropy production, σ. The expression for energy dissipation in the case of linearized Bénard convection is [7]

$$T\sigma = -\mathbf{T}_q . \nabla \ln T - \hat{\mathbf{P}}^{vs} : (\hat{\nabla}\mathbf{v})^s \tag{32}$$

and the linear laws in this case may be written as

$$\mathbf{T}_q = -L_\lambda \nabla \ln T, \tag{33}$$

$$\hat{\mathbf{P}}^{vs} = -L_s (\hat{\nabla}\mathbf{v})^s. \tag{34}$$

Thus the dissipation potentials are obtained in energy picture as

$$\psi^* = T\psi = \tfrac{1}{2}[L_\lambda (\nabla \ln T)^2 + L_s (\hat{\nabla}\mathbf{v})^s : (\hat{\nabla}\mathbf{v})^s], \tag{35}$$

$$\phi^* = T\phi = \tfrac{1}{2}[R_\lambda \mathbf{T}_q^2 + R_s \hat{\mathbf{P}}^{vs} : \hat{\mathbf{P}}^{vs}], \tag{36}$$

where

$$(\hat{\nabla}\mathbf{v})^s_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right), \quad (\alpha, \beta = 1, 2, 3). \tag{37}$$

We can, now, formulate the variational principle in actual form

$$\delta \int_V \left[-\mathbf{T}_q . \nabla \ln T - \hat{\mathbf{P}}^{vs} : (\hat{\nabla}\mathbf{v})^s - \frac{L_\lambda}{2} (\nabla \ln T)^2 - \frac{L_s}{2} \times (\hat{\nabla}\mathbf{v})^s : (\hat{\nabla}\mathbf{v})^s - \frac{R_\lambda}{2} \mathbf{T}_q^2 - \frac{R_s}{2} \hat{\mathbf{P}}^{vs} : \hat{\mathbf{P}}^{vs} \right] dV = 0. \tag{38}$$

Using the following vector identities

$$\nabla . (\mathbf{T}_q . \nabla \ln T) = \mathbf{T}_q . \nabla \ln T + \ln T \nabla . \mathbf{T}_q, \tag{39}$$

$$\nabla . (\mathbf{P}^v . \mathbf{v}) = \hat{\mathbf{P}}^{vs} : (\hat{\nabla}\mathbf{v})^s + \mathbf{v} . (\nabla . \mathbf{P}^v); \tag{40}$$

reduces the principle (38) to the form

$$\delta \int_V \left\{ -\ln T \left[c_v \rho_0 \frac{\partial \ln T}{\partial t} - c_v \rho_0 \beta \mathbf{n} . \mathbf{v} \right] - \mathbf{v} . \left[\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p - \rho_0 g \alpha \mathbf{n} \ln T \right] - \frac{L_\lambda}{2} (\nabla \ln T)^2 - \frac{L_s}{2} (\hat{\nabla}\mathbf{v})^s : (\hat{\nabla}\mathbf{v})^s - \frac{R_\lambda}{2} \mathbf{T}_q^2 - \frac{R_s}{2} \hat{\mathbf{P}}^{vs} : \hat{\mathbf{P}}^{vs} \right\} \times dV = 0. \tag{41}$$

In the above integral, we have neglected the terms containing the surface integrals, since in the problem under consideration the variation along the surface is neglected.

In order to derive the transport equations as the Euler–Lagrange equations of the above principle, we use the side conditions represented by the balance equations (30) and (31) in the following variational form

$$\ln T \delta \left[c_v \rho_0 \frac{\partial}{\partial t} \ln T + p \nabla . \mathbf{v} + \hat{\mathbf{P}}^{vs} : (\hat{\nabla}\mathbf{v})^s \right] = -\ln T \delta (\nabla . \mathbf{T}_q) = -\nabla . (\ln T \delta \mathbf{T}_q) + \nabla \ln T . \delta \mathbf{T}_q, \tag{42}$$

$$\mathbf{v} . \delta \left[\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p - g \alpha \mathbf{n} \ln T \right] = -\mathbf{v} . \delta [\nabla . \mathbf{P}^v] = -\nabla . (\mathbf{v} . \delta \mathbf{P}^v) + (\hat{\nabla}\mathbf{v})^s : \delta \hat{\mathbf{P}}^{vs}, \tag{43}$$

when the conductivities and resistances are constants, the following identities are valid

$$\tfrac{1}{2} L_\lambda \delta (\nabla \ln T)^2 = \nabla . [L_\lambda (\nabla \ln T) \delta \ln T] - \nabla . (L_\lambda \nabla \ln T) \delta \ln T, \tag{44}$$

$$\tfrac{1}{2} L_s \delta [(\hat{\nabla}\mathbf{v})^s : (\hat{\nabla}\mathbf{v})^s] = \nabla . [L_s (\hat{\nabla}\mathbf{v})^s : \delta \mathbf{v}] - \nabla . [L_s (\hat{\nabla}\mathbf{v})^s] . \delta \mathbf{v}. \tag{45}$$

Operating the principle (41) with δ and using the side conditions (42) and (43) and the identities (44) and (45), we get

$$\int_V \left\{ -\left[c_v \rho_0 \frac{\partial}{\partial t} \ln T - c_v \rho_0 \beta \mathbf{n} . \mathbf{v} - \nabla . (L_\lambda \nabla \ln T) \right] \delta \ln T - \left[\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p - \rho_0 g \alpha \mathbf{n} \ln T - \nabla . (L_s (\hat{\nabla}\mathbf{v})^s) \right] . \delta \mathbf{v} - (R_\lambda \mathbf{T}_q + \nabla \ln T) . \delta \mathbf{T}_q - (R_s \hat{\mathbf{P}}^{vs} + (\hat{\nabla}\mathbf{v})^s) : \delta \hat{\mathbf{P}}^{vs} \right\} \times dV = 0. \tag{46}$$

Here again we have neglected the terms containing the surface integrals. The vanishing of the volume integral in (46) gives the linearized equations of the Bénard convection

$$c_v\rho_0\frac{\partial T}{\partial t}-c_v\rho_0\beta\mathbf{n}.\mathbf{v}=\nabla.(L_\lambda\nabla\ln T),$$

or

$$\frac{\partial T}{\partial t}=\beta\mathbf{n}.\mathbf{v}+k\nabla^2T, \tag{47}$$

and

$$\rho_0\frac{\partial\mathbf{v}}{\partial t}+\nabla p-\rho_0g\alpha\mathbf{n}T=\nabla.[L_s(\mathring{\nabla}\mathbf{v})^s],$$

or

$$\frac{\partial\mathbf{v}}{\partial t}=-\frac{1}{\rho_0}\nabla p+g\alpha\mathbf{n}T+\nu\nabla^2\mathbf{v}, \tag{48}$$

here k and ν denote the heat diffusivity and the kinematic viscosity respectively.

4. THE APPLICATION OF THE DUAL FIELD METHOD

The variables $\ln T$ and $\mathbf{v}$ are the basic ones because their gradients are the driving forces of dissipative transport phenomena, therefore, we can introduce a second set of variables $\ln T^*$ and $\mathbf{v}^*$ which are related with the thermodynamic currents through the following constitutive laws

$$\mathbf{T}_q=-L_\lambda\nabla\ln T^*, \tag{49}$$

$$\mathring{\mathbf{P}}^{vs}=-L_s(\mathring{\nabla}\mathbf{v}^*)^s. \tag{50}$$

The assumed temperature and velocity fields $\ln T^*$ and $\mathbf{v}^*$ are to be determined with the help of balance equations (30) and (31) which take now the following form

$$c_v\rho_0\frac{\partial}{\partial t}\ln T-\nabla.(L_\lambda\nabla\ln T^*)=c_v\rho_0\beta\mathbf{n}.\mathbf{v}, \tag{51}$$

$$\rho_0\frac{\partial\mathbf{v}}{\partial t}-\nabla.[L_s(\mathring{\nabla}\mathbf{v}^*)^s]=-\nabla p+\rho_0g\alpha\mathbf{n}\ln T. \tag{52}$$

It is mentionable here that $\ln T^*$ and $\mathbf{v}^*$ are to satisfy the same boundary conditions as $\ln T$ and $\mathbf{v}$ respectively. The principle (41) now takes the following alternative form with the help of (49) and (50)

$$\delta\int_V\left\{-\ln T\left[c_v\rho_0\frac{\partial}{\partial t}\ln T-c_v\rho_0\beta\mathbf{n}.\mathbf{v}\right]-\frac{L_\lambda}{2}(\nabla\ln T)^2-\frac{L_\lambda}{2}(\nabla\ln T^*)^2-\mathbf{v}.\left[\rho_0\frac{\partial\mathbf{v}}{\partial t}+\nabla p-\rho_0g\alpha\mathbf{n}\ln T\right]-\frac{L_s}{2}(\mathring{\nabla}\mathbf{v})^s:(\mathring{\nabla}\mathbf{v})^s-\frac{L_s}{2}(\mathring{\nabla}\mathbf{v}^*)^s:(\mathring{\nabla}\mathbf{v}^*)^s\right\}\times\mathrm{d}V=0. \tag{53}$$

The partial integration reduces to the principle (53) in the following alternative form

$$\delta\int_V\left\{\ln T\left[c_v\rho_0\beta\mathbf{n}.\mathbf{v}-c_v\rho_0\frac{\partial}{\partial t}\ln T\right]-\frac{L_\lambda}{2}(\nabla\ln T)^2+\frac{L_\lambda}{2}\ln T^*\nabla^2\ln T^*+\mathbf{v}.\left[\rho_0g\alpha\mathbf{n}\ln T-\rho_0\frac{\partial\mathbf{v}}{\partial t}\right]-\frac{L_s}{2}(\mathring{\nabla}\mathbf{v})^s:(\mathring{\nabla}\mathbf{v})^s+\frac{L_s}{2}\mathbf{v}^*.\nabla^2\mathbf{v}^*\right\}\mathrm{d}V=0. \tag{54}$$

The pressure term vanishes from this volume integral due to the boundary conditions which can be seen by integrating partially. In evaluating the integral the conditions $\mathbf{v}^*=0$ and $\ln T^*=0$ at the boundaries are used. Substituting the values of $\nabla^2\ln T^*$ and $\nabla^2\mathbf{v}^*$ from (51) and (52), the principle (54) results as

$$\delta\int_V\left\{\ln T\left[c_v\rho_0\beta\mathbf{n}.\mathbf{v}-c_v\rho_0\frac{\partial}{\partial t}\ln T\right]-\frac{L_\lambda}{2}(\nabla\ln T)^2-\tfrac{1}{2}\ln T^*\left[c_v\rho_0\beta\mathbf{n}.\mathbf{v}-c_v\rho_0\frac{\partial}{\partial t}\ln T\right]+\mathbf{v}.\left[\rho_0g\alpha\mathbf{n}\ln T-\rho_0\frac{\partial\mathbf{v}}{\partial t}\right]-\frac{L_s}{2}(\mathring{\nabla}\mathbf{v})^s:(\mathring{\nabla}\mathbf{v})^s-\tfrac{1}{2}\mathbf{v}^*.\left[\rho_0g\alpha\mathbf{n}\ln T-\rho_0\frac{\partial\mathbf{v}}{\partial t}\right]\right\}\mathrm{d}V=0. \tag{55}$$

The pressure term again vanishes due to the boundary conditions.

The disturbances can be assumed as usual in the normal mode analysis:

$$\begin{aligned}\ln T&=\beta\,\mathrm{d}\theta(x_3)\cos\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_3&=\frac{\nu}{\mathrm{d}}G(x_3)\cos\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_1&=-\frac{a_1}{a^2}\nu\frac{\mathrm{d}G}{\mathrm{d}x_3}\sin\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_2&=-\frac{a_2}{a^2}\nu\frac{\mathrm{d}G}{\mathrm{d}x_3}\cos\frac{a_1x_1}{\mathrm{d}}\sin\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\end{aligned} \tag{56}$$

where $a=(a_1^2+a_2^2)^{1/2}$ is the wave number of the disturbance and ω is the frequency which, in principle, may be a complex quantity. The velocity components v_1, v_2 and v_3 satisfy the equation of continuity $\nabla.\mathbf{v}=0$. Similarly we can assume the disturbances $\ln T^*$ and $\mathbf{v}^*$

$$\begin{aligned}\ln T^*&=\beta\,\mathrm{d}\theta^*(x_3)\cos\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_3^*&=\frac{\nu}{\mathrm{d}}G^*(x_3)\cos\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_1^*&=-\frac{a_1}{a^2}\nu\frac{\mathrm{d}G^*}{\mathrm{d}x_3}\sin\frac{a_1x_1}{\mathrm{d}}\cos\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t},\\ v_2^*&=-\frac{a_2}{a^2}\nu\frac{\mathrm{d}G^*}{\mathrm{d}x_3}\cos\frac{a_1x_1}{\mathrm{d}}\sin\frac{a_2x_2}{\mathrm{d}}\mathrm{e}^{\varpi t}.\end{aligned} \tag{57}$$

The velocity components v^*, v^* and v^* satisfy the continuity equation $\nabla.\mathbf{v}^*=0$.

Substituting (56) and (57) into (55) and averaging over x_1–x_2 plane, we get the principle as

$$\delta\int_0^1\frac{d^3}{4}\left\{c_v\rho_0\nu\beta^2G\theta-\frac{\lambda\beta^2}{2}\left[\left(\frac{d\theta}{dz}\right)^2+a^2\theta^2\right]-\tfrac{1}{2}c_v\rho_0\nu\beta^2G\theta^*+\rho_0g\alpha\beta\nu G\theta-\rho_0\frac{\nu^3}{2d^4}\times\left[2\left(\frac{dG}{dz}\right)^2+a^2G^2+\frac{1}{a^2}\left(\frac{d^2G}{dz^2}\right)^2\right]-\tfrac{1}{2}\rho_0g\alpha\beta\nu G^*\theta-\omega\left[c_v\rho_0\beta^2d^2\theta^2+\rho_0\frac{\nu^2}{d^2}G^2+\rho_0\frac{\nu^2}{d^2a^2}\left(\frac{dG}{dz}\right)^2-\tfrac{1}{2}c_v\rho_0\beta^2d^2\theta\theta^*-\rho_0\times\frac{\nu^2}{d^2}GG^*-\rho_0\frac{\nu^2}{d^2a^2}\frac{dG}{dz}\frac{dG^*}{dz}\right]\right\}dz=0. \quad (58)$$

Here we have used the variable $z = x_3/d$. The boundary conditions in terms of G and θ are [9]

$$\begin{aligned} &z=0:\quad \theta=0,\ G=0,\ \frac{dG}{dz}=0;\\ &z=1:\quad \theta=0,\ G=0,\ \frac{dG}{dz}=0. \end{aligned} \quad (59)$$

The similar conditions are to be satisfied by θ^* and G^*. We use in (58), very simple trial functions for the disturbances, G and θ satisfying the boundary conditions (59)

$$G=A(z^2-2z^3+z^4), \quad (60)$$

$$\theta=B(z-z^2), \quad (61)$$

where A and B are the two variational parameters. The approximated disturbances θ^* and G^*, we get from the balance equations (51) and (52), which may be written as

$$\lambda\nabla^2\ln T^*=c_v\rho_0\frac{\partial}{\partial t}\ln T-c_v\rho_0\beta v_3, \quad (62)$$

$$\mu\nabla^4v_3^*=-\rho_0g\alpha\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\ln T+\rho_0\frac{\partial}{\partial t}\nabla^2v_3. \quad (63)$$

Using the expressions (56) and (57); (62) and (63) are obtained as

$$\left(\frac{d^2}{dz^2}-a^2\right)\theta^*=c_v\rho_0\frac{d^4}{\lambda}\omega\theta-\frac{c_v\rho_0\nu}{\lambda}G, \quad (64)$$

$$\left(\frac{d^4}{dz^4}-2a^2\frac{d^2}{dz^2}+a^4\right)G^*=\rho_0\frac{g\alpha\beta}{\nu\mu}a^2\theta+\frac{d^2}{\nu}\omega\times\left(\frac{d^2}{dz^2}-a^2\right)G. \quad (65)$$

For the linearized Bénard convection, the principle of exchange of stability is valid, i.e. ω is real [9]. Since the marginal stability curve is found by setting Real $\omega = 0$ and in this case ω is real, therefore, to get the marginal stability curve, we simply put $\omega = 0$ in the equations (64), (65) and in the principle (58), which reduce to:

$$\delta\int_0^1\frac{d^3}{4}\left\{c_v\rho_0\nu\beta^2G\theta-\frac{\lambda\beta}{2}\left[\left(\frac{d\theta}{dz}\right)^2+a^2\theta^2\right]-\tfrac{1}{2}c_v\rho_0\nu\beta^2G\theta^*-\rho_0\frac{\nu^3}{2d^4}\left[2\left(\frac{dG}{dz}\right)^2+a^2G^2+\frac{1}{a^2}\left(\frac{d^2G}{dz^2}\right)^2\right]+\rho_0g\alpha\beta\nu G\theta-\tfrac{1}{2}\rho_0g\alpha\beta\nu\theta G^*\right\}dz=0, \quad (66)$$

$$\left(\frac{d^2}{dz^2}-a^2\right)\theta^*=-\frac{c_v\rho_0\nu}{\lambda}G, \quad (67)$$

$$\left(\frac{d^4}{dz^4}-2a^2\frac{d^2}{dz^2}+a^4\right)G^*=g\alpha\frac{\beta d^4}{\nu^2}a^2\theta. \quad (68)$$

Solving for the functions θ^* and G^* from (67) and (68), with the help of (60) and (61), we get

$$\theta^*=A\frac{\nu}{k}\left[\frac{z^4}{a^2}-\frac{2z^3}{a^2}+\frac{(a^2+12)z^2}{a^4}-\frac{12z}{a^4}+\frac{(2a^2+24)}{a^6}+\alpha_1e^{az}+\alpha_2e^{-az}\right], \quad (69)$$

$$G^*=\frac{B}{a^2}\frac{g\alpha\beta d^4}{\nu^2}\left[-\frac{4}{a^2}+z-z^2+(\alpha_3z+\alpha_4)e^{az}+(\alpha_5z+\alpha_6)e^{-az}\right], \quad (70)$$

where

$$\alpha_1=\frac{2(a^2+12)}{a^6}\frac{e^{-a}-1}{e^a-e^{-a}},\qquad \alpha_2=\frac{2(a^2+12)}{a^6}\frac{1-e^a}{e^a-e^{-a}},$$

$$\alpha_3=\frac{d_3b_1-d_1b_3}{d_1b_2-d_2b_1},\qquad \alpha_4=\frac{d_3b_1-d_1b_3}{d_2b_1-d_1b_2},$$

$$\alpha_5=\frac{4}{a}-1-2a\alpha_4-\alpha_3,\qquad \alpha_6=\frac{4}{a^2}-\alpha_4,$$

$$b_1=(1+a)e^{2a}+a-1,\qquad b_2=ae^{2a}+2a^2-a,$$

$$b_3=e^a-a+5.\qquad d_1=e^{2a}-1,$$

$$d_2=e^{2a}-2a+1,\qquad d_3=\frac{4}{a^2}(e^a-1)+1-\frac{4}{a}.$$

The expressions (69) and (70) satisfy the boundary conditions

$$\begin{aligned} &z=0:\ G^*=0,\quad \frac{dG^*}{dz}=0,\quad \theta^*=0,\\ &z=1:\ G^*=0,\quad \frac{dG^*}{dz}=0,\quad \theta^*=0. \end{aligned} \quad (71)$$

Finally the principle (66)—with the substitution of the expressions for θ, G, θ^* and G^* and integration—reduces to

$$\delta\left[\frac{d^3}{4}\left\{c_v\rho_0\nu\beta^2\frac{AB}{140}-\frac{\lambda\beta^2}{2}\left(\frac{1}{3}+\frac{a^2}{30}\right)B^2-\frac{1}{2\lambda}(c_v\rho_0\nu\beta)^2\beta_1A^2+\rho_0\frac{k\nu^2}{d^4}R\frac{AB}{140}-\frac{\rho\nu^3}{2d^4}\times\left(\frac{4}{105}+\frac{a^2}{630}+\frac{4}{5a^2}\right)A^2-\tfrac{1}{2}\rho_0\frac{k^2\nu^2}{d^4}\times R^2\beta_2\frac{B^2}{a^2}\right\}\right]=0; \quad (72)$$

where

$$\beta_1 = \frac{1}{a^4}\left(\frac{4}{5a^2} + \frac{a^2}{630} - \frac{2}{105}\right) + \alpha_1(c_7 - 2c_5 + c_1) + \alpha_2(c_8 - 2c_6 + c_4),$$

$$\beta_2 = \frac{1}{30} - \frac{2}{3a^2} + \alpha_3(c_3 - c_5) + \alpha_4(c_1 - c_3) + \alpha_5(c_4 - c_6) + \alpha_6(c_2 - c_4),$$

$$c_1 = \int_0^1 z\,e^{az}\,dz, \qquad c_2 = \int_0^1 z\,e^{-az}\,dz,$$

$$c_3 = \int_0^1 z^2\,e^{az}\,dz, \qquad c_4 = \int_0^1 z^2\,e^{-az}\,dz,$$

$$c_5 = \int_0^1 z^3\,e^{az}\,dz, \qquad c_6 = \int_0^1 z^3\,e^{-az}\,dz,$$

$$c_7 = \int_0^1 z^4\,e^{az}\,dz, \qquad c_8 = \int_0^1 z^4\,e^{-az}\,dz,$$

$R = g\alpha\beta d^4/kv$ is the non-dimensional parameter called Rayleigh number.

From the principle (72), we can get three different expressions for Rayleigh number according to the three representations of the Gyarmati's principle.

5. RESULTS

(i) *Universal representation*

In this case, we vary the principle (72) with respect to the variational parameter B and the Rayleigh number, R, to get the neutral stability curve, that is; a relation between the wave number, a, and the Rayleigh number, R. The neutral stability curve is found to be

$$R = a^2(10 + a^2)/30\beta_2. \tag{73}$$

The critical wave number, a_c, and the corresponding Rayleigh number, that is; critical Rayleigh number, R_c, were found numerically

$$\text{critical wave number, } a_c = 3{\cdot}122,$$

$$\text{critical Rayleigh number, } R_c = 1748{\cdot}696.$$

These values are remarkably close to the exact values 3·117 and 1708 obtained by Chandrasekhar [9].

(ii) *Force representation*

In the force representation the principle is varied with respect to the thermodynamical forces only. Therefore, in this case the terms which are due to the thermodynamical currents have no contribution and the principle (72) takes the simpler form

$$\delta\left[\frac{d^3}{4}\left\{c_v\rho_0\nu\beta^2\frac{\mathbf{AB}}{140} - \frac{\lambda\beta^2}{2}\left(\frac{1}{3} + \frac{a^2}{30}\right)B^2 + \rho_0\frac{k\nu^2}{d^4} \times \frac{R\mathbf{AB}}{140} - \rho_0\frac{\nu^3}{2d^4}\left(\frac{4}{105} + \frac{a^2}{630} + \frac{4}{5a^2}\right) \times A^2\right\}\right] = 0. \tag{74}$$

In the above principle the underlined parameters are the contributions due to currents and we do not vary with respect to these quantities. Thus varying the principle (74) with respect to variational parameters A and B, we get the following equation for the neutral stability curve

$$R = (140)^2\left(\frac{1}{3} + \frac{a^2}{30}\right)\left(\frac{4}{105} + \frac{a^2}{630} + \frac{4}{5a^2}\right). \tag{75}$$

This expression is just similar to that found by Schechter [10] by local potential method. As it has already been proved by Gyarmati and others [5–7] that local potential method is in practical respect equivalent to the force representation of the governing principle of dissipative processes, this practical problem confirms the theoretically proved equivalency. In this case we get the critical wave number, a_c, and critical Rayleigh number, R_c, as

$$a_c = 3{\cdot}117,$$

$$R_c = 1749{\cdot}976.$$

(iii) *Flux representation*

The characteristic of this representation lies in the fact that the governing principle of Gyarmati is varied with respect to the current densities, keeping the force terms constant. Therefore the principle reduces to

$$\delta\left[\frac{d^3}{4}\left\{c_v\rho_0\beta^2\frac{\mathbf{AB}}{140} - \frac{1}{2}\frac{(c_v\rho_0\nu\beta)^2}{\lambda}\beta_1A^2 + \rho_0\frac{k\nu^2}{d^4}R\frac{A\mathbf{B}}{140} - \tfrac{1}{2}\rho_0\frac{k^2\nu}{d^4}\frac{R^2}{a^2}\beta_2\mathbf{B}^2\right\}\right] = 0. \tag{76}$$

Varying the principle (76) with respect to the variational parameters **A** and **B**, we get the neutral stability curve

$$R = a^2/(140)^2\beta_1\beta_2. \tag{77}$$

The relation (77) was solved numerically and the critical wave number, a_c, and the corresponding Rayleigh number, R_c, are found to be

$$a_c = 3{\cdot}119,$$

$$R_c = 1706{\cdot}815.$$

These values are quite in agreement with those of exact results [9].

6. CONCLUSION

The critical wave numbers and Rayleigh numbers obtained with the help of the governing principle in universal, force and flux representations are listed in Table 1 with those of exact values.

Table 1

	Exact	Universal	Force	Flux
Critical wave number, a_c	3·117	3·122	3·117	3·119
Critical Rayleigh number, R_c	1707·762	1748·696	1749·976	1706·815

It is interesting to note that the values of critical wave numbers and critical Rayleigh numbers are remarkably close to the more precise results obtained by Chandrasekhar [9]. In particular, the critical

Rayleigh number obtained by flux representation method differs with that of exact result by less than 0·05 per cent, though the result is obtained on one term approximation of velocity and temperature profiles. The results obtained by universal representation are quite satisfactory for the most of engineering applications. The results can be improved by considering more terms in the Fourier series in terms of which the temperature and velocity perturbations can be expressed.

It may be mentioned that the results found for the Bénard convection in universal, force and flux representations are in the same order as found by Stark for one-dimensional heat conduction problem [8] with the help of governing principle of dissipative processes. The flux representation gives the best approximation, while universal form comes next to it and force stands the last. It may be noted that the universal form of the principle is the actual variational principle and the results obtained on the basis of this actual form differ by less than 2·5 per cent with that of exact values in this fundamental problem of stability. Thus the applicability of the governing principle is established for the solution of the practical problems of hydrodynamic stability.

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REFERENCES

1. I. Gyarmati, *Z. Phys. Himii (Moscow)* **39**, 1489 (1965).
2. I. Gyarmati, *Acta Chim. Hung.* **43**, 353 (1965).
3. L. Onsager, Reciprocal relations in irreversible processes—I, *Phys. Rev.* **37**, 405 (1931).
4. L. Onsager, Reciprocal relations in irreversible processes—II, *Phys. Rev.* **38**, 2265 (1931).
5. I. Gyarmati, On the governing principle of dissipative processes and its extension to non-linear problems, *Ann. Phys.* **23**, 353 (1969).
6. I. Gyarmati, *Non-Equilibrium Thermodynamics, Field Theory and Variational Principles*. Springer, Berlin (1970).
7. P. Singh, The applications of the governing principle of dissipative processes to thermohydrodynamical stability, D.Sc. Dissertation, Hungarian Academy of Sciences, Budapest (1973).
8. A. Stark, Approximation methods for the solution of heat conduction problems using Gyarmati's principle, *Ann. Phys.* **7**, 53 (1974).
9. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford (1961).
10. P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations*. Wiley–Interscience, New York (1971).

APPLICATION DES LOIS FONDAMENTALES DES PROCESSUS DISSIPATIFS A LA CONVECTION DE BENARD

Résumé—L'article traite de l'application des lois fondamentales des processus dissipatifs à la convection de Bénard, c'est à dire au mouvement provoqué par l'instabilité thermique d'une mince couche horizontale de fluide soumise à une différence de température constante. Les équations linéarisées de la convection de Bénard sont obtenues à l'aide de la forme universelle des lois fondamentales. On a utilisé la méthode dite du champ dual développée sur la base des lois fondamentales des processus dissipatifs afin d'obtenir une solution du problème dans chacune des trois présentations du principe, c'est à dire respectivement en termes universels, de force et de flux. Les résultats obtenus à l'aide du principe variationnel lui-même, c'est à dire par la forme universelle du principe, différent de moins de 2,5% des valeurs exactes alors que la représentation en termes de flux fournit des résultats mieux approchés et que la représentation en termes de force s'avère la plus mauvaise.

DIE ANWENDUNG DES HERRSCHENDEN PRINZIPS DISSIPATIVER PROZESSE AUF DIE BÉNARD-KONVEKTION

Zusammenfassung—Die Arbeit befaßt sich mit der Anwendung des "Governing Principle of Dissipative Processes" (Herrschendes Prinzip dissipativer Prozesse), kurz GPDP, auf die Bénard-Konvektion, das heißt auf die Bewegung, die durch die thermische Instabilität einer dünnen horizontalen Flüssigkeitsschicht hervorgerufen wird, wenn ein konstanter Temperaturunterschied aufgeprägt wird. Die linearisierten Gleichungen der Bénard-Konvektion werden mit Hilfe der allgemeinen Form des "Governing Principle" (herrschenden Prinzips) gewonnen. Die sogenannte Dualfeldmethode auf der Basis von GPDP wird angewandt, um die Lösung des Problems in allen drei Darstellungen des Prinzips, das heißt allgemein, beziehungsweise mit Kräften oder Flüssen zu erhalten. Es wurde gefunden, daß die Ergebnisse, die durch das Variationsprinzip, das heißt durch die allgemeine Form des Prinzips erhalten wurde, um weniger als 2,5% von den exakten Werten abweichen, während die Darstellung mit Flüssen die besten Näherungsergebnisse und die mit Kräften die schlechtesten Resultate liefern.

ПРИМЕНЕНИЕ ОСНОВНОГО ДИССИПАТИВНОГО ПРИНЦИПА К КОНВЕКЦИИ БЕНАРА

Аннотация — В статье рассматривается применение основного принципа диссипативных процессов к конвекции Бенара, т. е. движению, возникающему из-за конвективной неустойчивости тонкого горизонтального слоя жидкости при наличии поперек слоя постоянной разности температур. Линеаризованные уравнения конвективного движения Бенара выводятся с помощью основного принципа в его универсальном виде. Для решения задачи используется так называемый двухполевой метод, разработанный на основе указанного принципа. Принцип используется в трех формах: в универсальном виде, для сил и для потоков.

Найдено, что результаты, полученные с помощью истинного вариационного принципа, т. е. с помощью принципа в его универсальном виде, примерно на 2,5% отличаются от точных значений. Наилучшее приближение достигается при использовании принципа для потоков, в то время как при использовании принципа для сил приближение самое плохое.